



TITLE:

On some periodic modules for group algebras of finite groups(Representation Theory of Finite Groups and Algebras)

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CITATION:

Hida, Akihiko. On some periodic modules for group algebras of finite groups(Representation Theory of Finite Groups and Algebras). 数理解析研究所講究録 1994, 877: 14-18

ISSUE DATE:

1994-06

URL:

<http://hdl.handle.net/2433/84159>

RIGHT:

On some periodic modules for group algebras of finite groups

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1. Introduction

Let G be a finite group and let k be a field of characteristic $p > 0$. Let M be a finitely generated kG -module. Let $\phi : P \rightarrow M$ be the projective cover of M and let $\Omega(M)$ be the kernel of ϕ . We define inductively as $\Omega^{n+1}(M) = \Omega(\Omega^n(M))$ for any positive integer n . Similarly we define $\Omega^n(M)$ for a negative integer n using the injective hull. We say that M is periodic if $\Omega^n(M) \cong M$ for some $n \geq 1$. If n is the smallest such integer then n is called the period of M .

Let

$$G = G(m, n) = \langle s, t \mid s^{p^n} = t^{p^m} = 1, s^{-1}ts = t^{p^{m-n}+1} \rangle$$

be a metacyclic p -group of order p^{m+n} where p is an odd prime and $m - n > 0, n > 0$. The cohomology ring $H^*(G, k)$ was determined by Diethelm [7]. We shall follow the notation in [7]. By [7, Theorem 2],

$$\begin{aligned} (1.1) \quad H^*(G, k) &= k[a_1, \dots, a_{p-1}, b, y, v, w] \\ a_i a_j &= a_i y = a_i v = b^2 = v^2 = 0, \\ \deg a_i &= 2i - 1, \deg b = 1, \deg y = 2, \\ \deg v &= 2p - 1, \deg w = 2p, \\ b, y &\in \text{Im}(\inf : H^*(G / \langle t \rangle, k) \rightarrow H^*(G, k)), \\ \text{res}_{\langle t \rangle}^G(a_1) &\neq 0. \end{aligned}$$

(We omit all relations which are consequences of the skew commutative relation.)

Let $\hat{y}^i : \Omega^{2i}(k) \rightarrow k$ be the cocycle which represents y^i and let L_i be the kernel of \hat{y}^i for $i \geq 1$. Then w generates the periodicity of L_i since $H^*(G, k)$ is finitely generated over $k[y, w]$ as a module (cf. [1, 5.10]). Moreover, by [2, Lemma 4.4] and [4, Lemma 4.1], we have the following.

(1.2) For every $i \geq 1$, L_i is an indecomposable periodic kG -module with period 2 or $2p$.

In [8], Okuyama and Sasaki showed that the period of L_p is exactly $2p$. The following is our main result.

THEOREM. The period of L_i is $2p$ if $i \geq 2$ and 2 if $i = 1$.

Let G be an arbitrary finite group. For a kG -module M , we set

$$\begin{aligned}\hat{H}^i(G, M) &= \underline{Hom}_{kG}(\Omega^i(k), M) \\ &= Hom_{kG}(\Omega^i(k), M) / PHom_{kG}(\Omega^i(k), M)\end{aligned}$$

where $PHom_{kG}(\Omega^i(k), M)$ is a subspace of $Hom_{kG}(\Omega^i(k), M)$ generated by projective homomorphisms. If N is a kG -module, there exists a product

$$\hat{H}^i(G, M) \otimes \hat{H}^j(G, N) \longrightarrow \hat{H}^{i+j}(G, M \otimes N).$$

In particular we have the Tate duality, namely,

$$\hat{H}^i(G, k) \otimes \hat{H}^{-(i+1)}(G, k) \longrightarrow \hat{H}^{-1}(G, k) = k$$

is non-degenerate for any i (cf.[6,XII]).

Let $\zeta (\neq 0) \in H^i(G, k) (i > 0)$. Then ζ is represented by $\hat{\zeta} : \Omega^i(k) \longrightarrow k$. We set $L_\zeta = Ker \hat{\zeta}$. By definition of L_ζ there exists an exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^i(k) \xrightarrow{\hat{\zeta}} k \longrightarrow 0.$$

Hence we have a long exact sequence

$$\begin{aligned}(1.3) \quad & \longrightarrow \hat{H}^{j-1}(G, k) \xrightarrow{\delta} \hat{H}^j(G, L_\zeta) \longrightarrow \\ & \hat{H}^j(G, \Omega^i(k)) \cong \hat{H}^{j-i}(G, k) \xrightarrow{\zeta} \hat{H}^j(G, k) \longrightarrow .\end{aligned}$$

Remark 1.4. If H is a subgroup of G and if $p || |H|$, then the transfer map

$$t_H^G : \hat{H}^{-1}(H, k) \longrightarrow \hat{H}^{-1}(G, k)$$

is not zero. Indeed, consider the exact sequence

$$0 \longrightarrow \Omega(k) \xrightarrow{\iota} P_0 \xrightarrow{\epsilon} k \longrightarrow 0$$

where $P_0 \xrightarrow{\epsilon} k$ is the projective cover of k as a kG -module. Let $f (\neq 0) \in \hat{H}^{-1}(G, k) = Hom_{kG}(k, \Omega(k))$. Since P_0 is projective there exists $g \in Hom_{kH}(k, P_0)$ such that $\iota \circ f = Tr_H^G(g)$. Since $\epsilon \circ g = 0$, $g = \iota \circ g'$ for some $g' \in Hom_{kH}(k, \Omega(k))$. Then $\iota \circ Tr_H^G(g') = Tr(g) = \iota \circ f$ and so $Tr_H^G(g') = f$.

2.Proof of Theorem

In this section, we assume that $G = G(m, n)$ where p is odd and $m - n > 0, n > 0$. We take the following k -basis of $\hat{H}^i(G, k) (i = 1, 2)$ (cf.(1.1))

$$\begin{aligned}\hat{H}^1(G, k) &: a_1, b \\ \hat{H}^2(G, k) &: a_1 b, y\end{aligned}$$

and the dual basis with respect to the Tate duality,

$$\begin{aligned}\hat{H}^{-2}(G, k) &: (a_1)^*, b^* \\ \hat{H}^{-3}(G, k) &: (a_1 b)^*, y^*.\end{aligned}$$

First we consider the period of L_1 . We set $H = \langle s, z = [s, t] \rangle \triangleleft G$. Then by [7, Theorem 1],

$$\begin{aligned}(2.1) \quad H^*(H, k) &= k[a', b', x', y'] \\ \deg a' &= \deg b' = 1, \deg x' = \deg y' = 2, \\ \text{res}_{\langle z \rangle}^H(a') &\neq 0, b' = \text{res}_H^G(b), y' = \text{res}_H^G(y).\end{aligned}$$

LEMMA 2.2. $\text{res}_H^G(a_1^*) \neq 0$. In particular, $\text{res}_H^G((a_1 b)^*) \neq 0$.

PROOF: Since $\text{res}_{\langle t \rangle}^G(a_1) \neq 0$, $a_1^* = t_{\langle t \rangle}^G(c)$ for some $c \in \hat{H}^{-2}(\langle t \rangle, k)$ where $t_{\langle t \rangle}^G$ is the transfer map (cf. Remark 1.4). Hence $\text{res}_H^G(a_1^*) = t_{\langle z \rangle}^H(\text{res}_{\langle z \rangle}^{\langle t \rangle}(c)) \neq 0$. Since $b(a_1 b)^* = a_1^*$, it follows that $\text{res}_H^G((a_1 b)^*) \neq 0$.

LEMMA 2.3. There exists $\zeta \in H^2(H, k)$ such that $\zeta \text{res}_H^G((a_1 b)^*) = 0$ and $L_1 \otimes L_\zeta$ is a projective kH -module.

PROOF: Since $y(a_1 b)^* = 0$ and $\text{res}_H^G(b(a_1 b)^*) \neq 0$, some k -linear combination of $a'b'$ and x' satisfies the condition of Lemma.

Now consider the following commutative diagram (cf.(1.3))

$$\begin{array}{ccccc}\hat{H}^{-1}(G, L_1) & \longrightarrow & \hat{H}^{-1}(G, \Omega^2(k)) & \xrightarrow{y} & \hat{H}^{-1}(G, k) \\ \text{res} \downarrow & & \downarrow \text{res} & & \\ \hat{H}^{-1}(H, L_1) & \longrightarrow & \hat{H}^{-1}(H, \Omega^2(k)) & & \\ \zeta \downarrow & & \downarrow \zeta & & \\ \hat{H}^0(H, k) & \xrightarrow{\delta} & \hat{H}^1(H, L_1) & \longrightarrow & \hat{H}^1(H, \Omega^2(k)).\end{array}$$

Then there exists $e \in \hat{H}^{-1}(G, L_1)$ such that $\zeta \text{res}_H^G(e) = \delta(1)$. Note that $\zeta : \hat{H}^{-1}(H, L_1) \longrightarrow \hat{H}^1(H, L_1)$ is an isomorphism. Let $\theta = \delta(1) \in \hat{H}^1(G, L_1) \cong \text{Hom}_{kG}(k, \Omega^{-1}(L_1))$.

Then we have the following commutative diagram,

$$\begin{array}{ccc}
Hom_{kG}(\Omega^{-1}(L_1), \Omega(L_1)) & \xrightarrow{\theta^*} & Hom_{kG}(k, \Omega(L_1)) = \hat{H}^{-1}(G, L_1) \\
\downarrow res & & \downarrow res \\
\underline{Hom}_{kH}(\Omega^{-1}(L_1), \Omega(L_1)) & \longrightarrow & \underline{Hom}_{kH}(k, \Omega(L_1)) = \hat{H}^{-1}(H, L_1) \\
\downarrow \zeta & & \downarrow \zeta \\
\underline{Hom}_{kH}(\Omega^{-1}(L_1), \Omega^{-1}(L_1)) & \longrightarrow & \underline{Hom}_{kH}(k, \Omega^{-1}(L_1)) = \hat{H}^1(H, L_1).
\end{array}$$

Since y annihilates $Ext_{kG}(L_1, L_1)$ by [5, Theorem 4.1] (or [1, Proposition 5.9.6]) θ^* is onto. Hence $e = \theta^*(f)$ for some $f \in Hom_{kG}(\Omega^{-1}(L_1), \Omega(L_1))$ and $\zeta res_H^G(f)$ is an isomorphism (modulo projective). Hence f is an isomorphism.

Next we consider the case $i \geq 2$. We shall show that

$$(2.4) \quad res_H^G(y^{i-1} \hat{H}^{-1}(G, L_i)) = 0$$

but

$$res_H^G(y^{i-1} \hat{H}^1(G, L_i)) \neq 0.$$

LEMMA 2.5. Suppose that $i \geq 2$. If $c \in \hat{H}^{-(2i+1)}(G, k)$ and $y^i c = 0$ then $y^{i-1} c = 0$.

Hence in the following commutative diagram

$$\begin{array}{ccccc}
\hat{H}^{-1}(G, L_i) & \longrightarrow & \hat{H}^{-(2i+1)}(G, k) & \xrightarrow{y^i} & \\
y^{i-1} \downarrow & & y^{i-1} \downarrow & & \\
\delta \longrightarrow \hat{H}^{2i-3}(G, L_i) & \longrightarrow & \hat{H}^{-3}(G, k) & &
\end{array}$$

we have $y^{i-1} \hat{H}^{-1}(G, L_i) \subseteq Im \delta$. Now consider the commutative diagram,

$$\begin{array}{ccccc}
\hat{H}^{2i-4}(G, k) & \xrightarrow{\delta} & H^{2i-3}(G, L_i) & & \\
y \downarrow & & y \downarrow & & \\
\hat{H}^{-2}(G, k) & \xrightarrow{y^i} \hat{H}^{2i-2}(G, k) & \xrightarrow{\delta'} & H^{2i-1}(G, L_i). &
\end{array}$$

Since $y^i \hat{H}^{-2}(G, k) = 0$ (cf. [3, Lemma 2.2]) δ' is monomorphism. If $c = \delta(h) \in y^{i-1} \hat{H}^{-1}(G, L_i)$ ($h \in \hat{H}^{2i-4}(G, k)$) then $yc = 0$ since y^i annihilates $Ext_{kG}^*(L_i, L_i)$ ([5, Theorem 4.1] or [2, Proposition 5.9.6]). Hence we have $yh = 0$. Since $res_H^G(y)$ is not a zero divisor in $H^*(H, k)$ (cf. (2.1)) we have that $res_H^G(h) = 0$ and $res_H^G(c) = 0$.

Next consider the following commutative diagram,

$$\begin{array}{ccc}
 H^0(G, k) & \xrightarrow{\delta} & H^1(G, L_i) \\
 \text{res} \downarrow & & \text{res} \downarrow \\
 H^0(H, k) & \longrightarrow & H^1(H, L_i) \\
 y^{i-1} \downarrow & & y^{i-1} \downarrow \\
 \xrightarrow{y^i} H^{2(i-1)}(H, k) & \xrightarrow{\delta''} & H^{2i-1}(H, L_i)
 \end{array}$$

Since δ'' is monomorphism we have $\text{res}(y^{i-1}\delta(1)) \neq 0$.

Actually, to prove that $\Omega^2(L_i) \not\cong L_i$ for $i \geq 2$, it suffices to consider only the case $i = 2$ by the following Proposition.

PROPOSITION 2.7. *Let M be a non-projective indecomposable kG -module. Suppose that y^i annihilates $\text{Ext}_{kG}^*(M, M)$. If $\Omega^2(L_i) \cong L_i$ then $\Omega^2(M) \cong M$.*

PROOF: By [5, Lemma 4.4] (or [1, Proposition 5.9.5]), $L_i \otimes M \cong \Omega(M) \oplus \Omega^{2i}(M) \oplus (\text{proj})$. So the result follows.

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